Mansour @FOCS

Tuesday, December 7, 2021 16:45

Sharper bounds on the Fourier concentration of DNFs

aka

HOW SIMPLE ARE DNFs ?

joint with Li-Yang Tan

PART I: FOURIER CONCENTRATION

Boolean analysis

- $\cancel{f : \{0,1\}^n \to \{0,1\}}$

  $\{+1,-1\}^n \to \{+1,-1\}$

- Can (nicely) represent f as a multilinear polynomial.

e.g. $\mathrm{MAJ}(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_1 x_2 x_3$

$\widehat{\mathrm{MAJ}}(\{1\}) = \frac{1}{2}$ $\qquad$ $\widehat{\mathrm{MAJ}}(\{1,2,3\}) = -\frac{1}{2}$

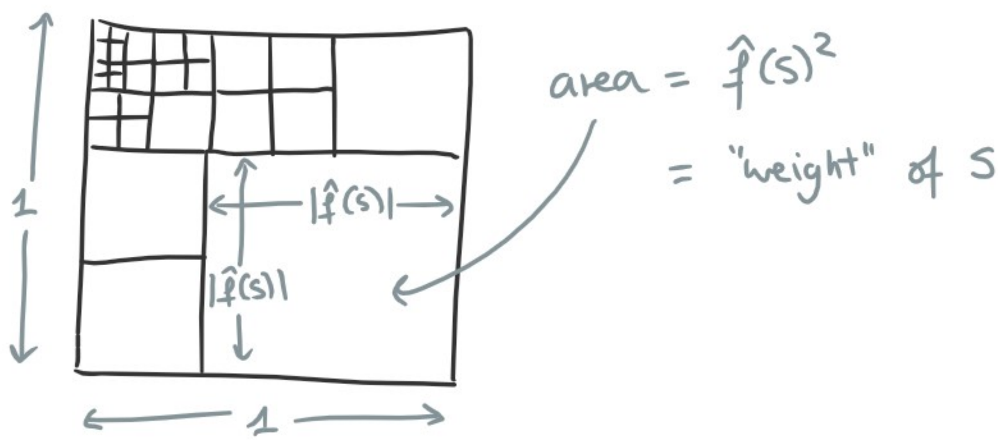
- For $S \subseteq [n]$, let $\hat{f}(S)$ be the coefficient of $\prod_{i \in S} x_i$.

Fourier coefficients

- We call those $\hat{f}(S)$ the Fourier coefficients of f.
- They reveal properties of f.

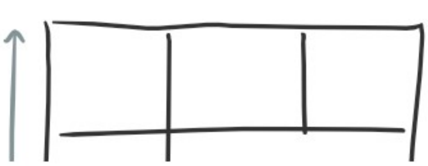
  e.g. bias, influence, sensitivity

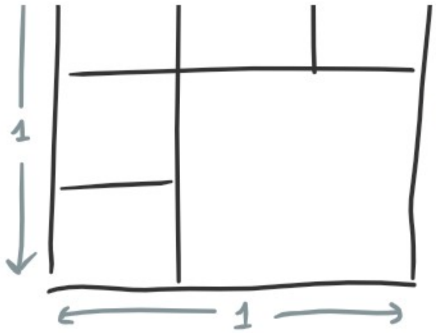
- $\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1 \Rightarrow$ "pieces of a unit square"



Fourier sparsity

- When few coefficients are nonzero, f is sparse.





only 6 nonzero coefficients

$\Rightarrow$ "6-sparse"

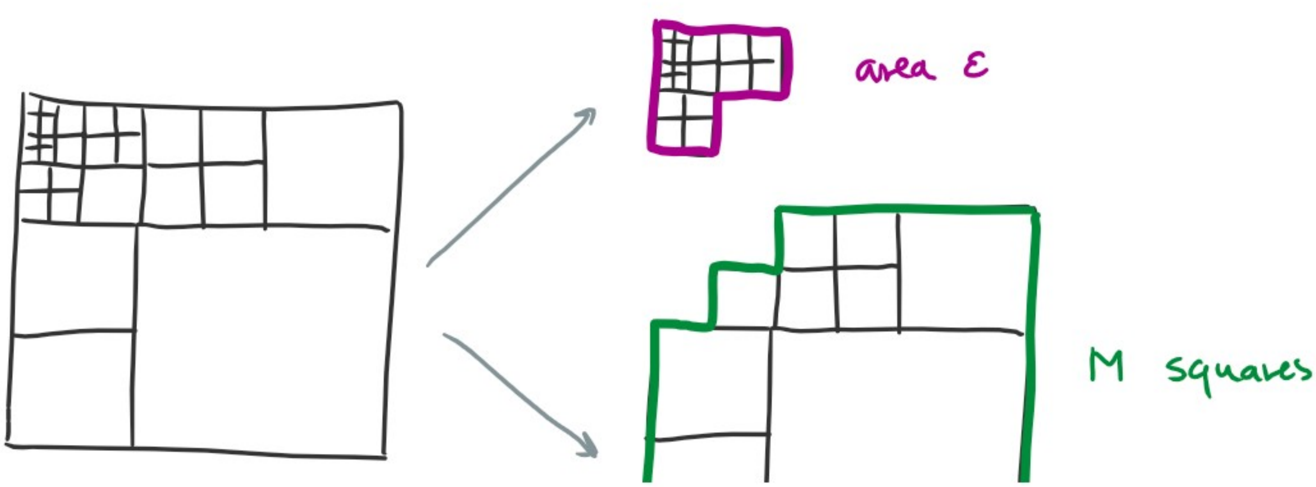
- Sparse functions are "simple".

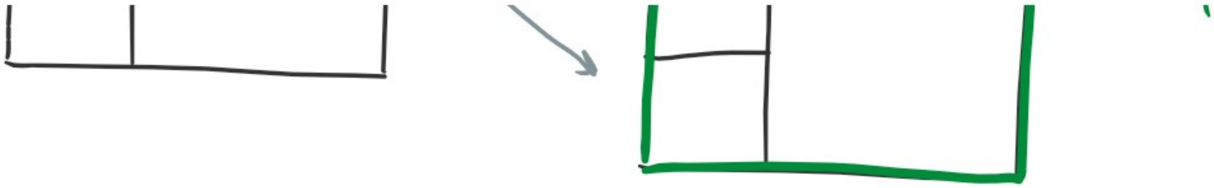
  e.g. easy to learn from queries

- But most functions <u>not</u> sparse. :(

Fourier concentration

- Almost as good: concentrated on few coefficients.





"f is ε-concentrated on M coefficients"

- Equivalently: can ε-approximate by an M-sparse function.
- Most successful notion of simplicity:
  - flexible enough to apply to many functions
  - strong enough to give nice properties

PART II : DNFs

DNFs

DNF = OR of ANDs

$$f = \underbrace{\overline{x_1}}_{T_1} \vee \underbrace{(x_2 \wedge \overline{x_3})}_{T_2} \vee \underbrace{(x_3 \wedge x_4)}_{T_3} \vee \underbrace{(x_3 \wedge \overline{x_5})}_{T_4}$$

f	typical
4	poly(n)

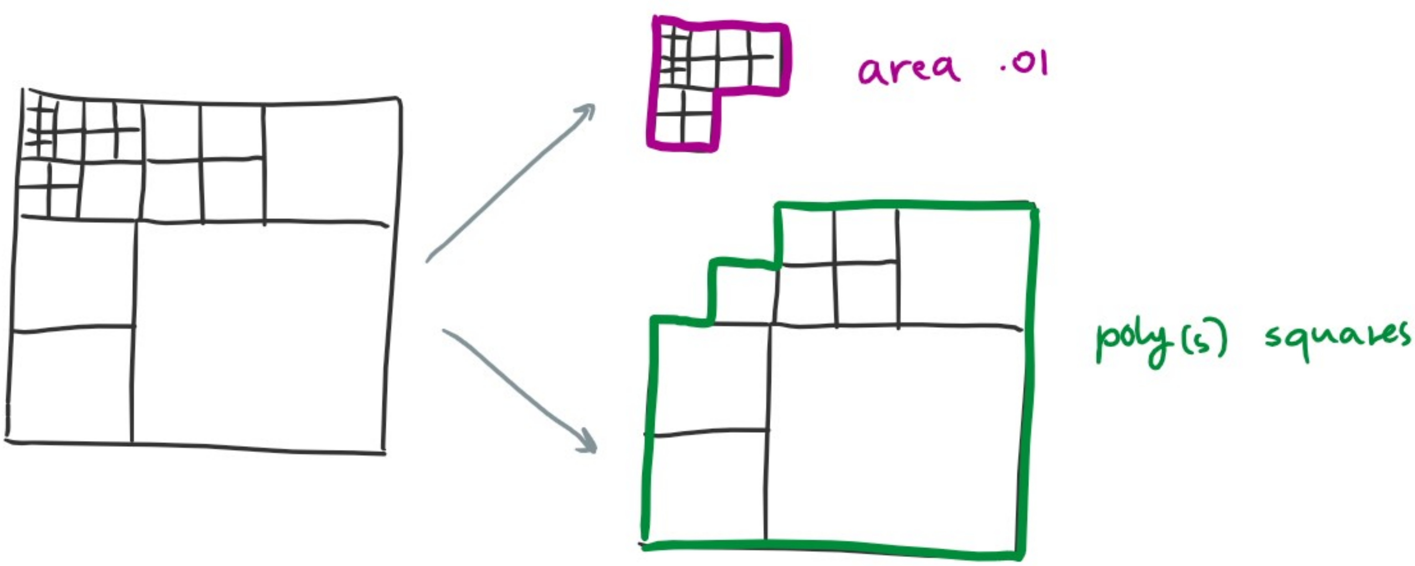
- size s = # terms

• read k = # of occurences of a variable   3   log n

Note : $k \leq s$.

Mansour's conjecture

Every size-s DNF .01 - concentrated on poly(s) coefficients ?



What's known : $s^{\log\log s}$

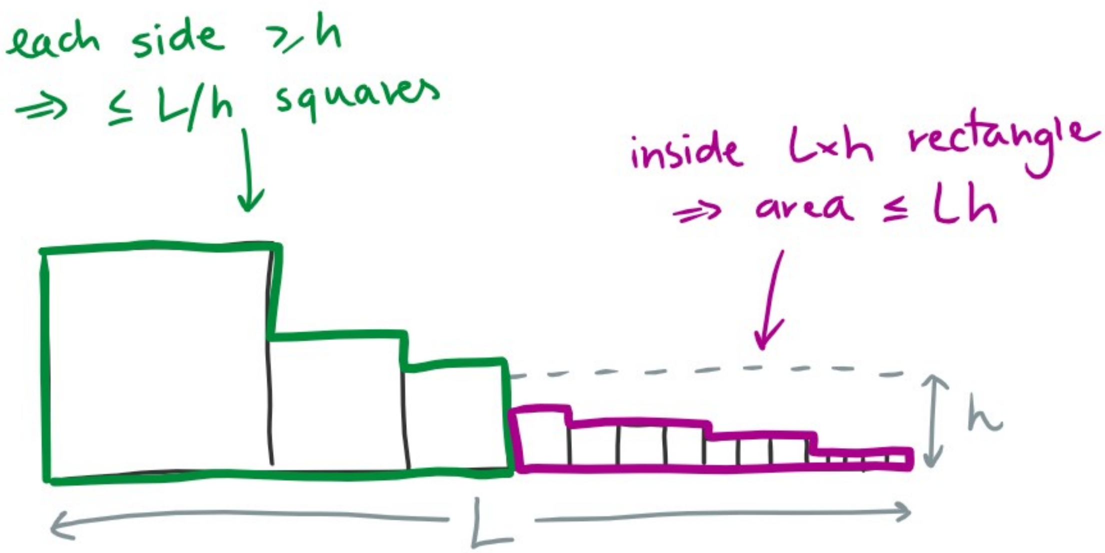
We show : • $s^{\log\log k}$ (since $k \leq s$, always better)

• poly(s) when $k \leq \tilde{\Theta}(\log\log s)$

PART III : MANSOUR'S "TOTAL LENGTH" METHOD

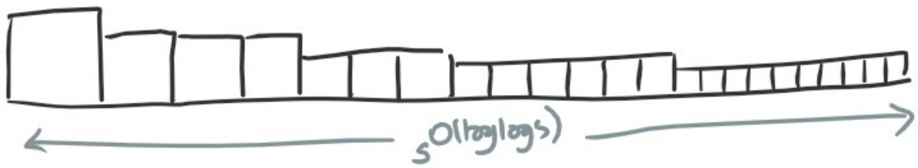
Plan : bound the "total length"

Say $\sum |\hat{f}(S)| \leq L$.



Set $h = \frac{.01}{L}$ $\Rightarrow$ .01 - concentrated on $100L^2$ coefficients.

Mansour's theorem : $\sum |\hat{f}(S)| \le s^{O(\log\log s)}$



- So .01 - concentrated on $s^{O(\log\log s)}$ coefficients.
- But $s^{O(\log\log s)}$ tight, even for very "simple" DNFs ! :(

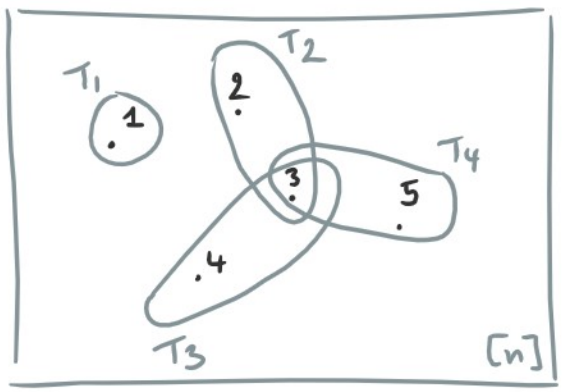
e.g. TRIBES $= (x_1 \wedge \cdots \wedge x_w) \vee (x_{w+1} \wedge \cdots \wedge x_{2w}) \vee \cdots \vee (x_{n-w+1} \wedge \cdots \wedge x_n)$

(read $k=1$)

PART IV : THE NEW IDEA : TERM COVERS

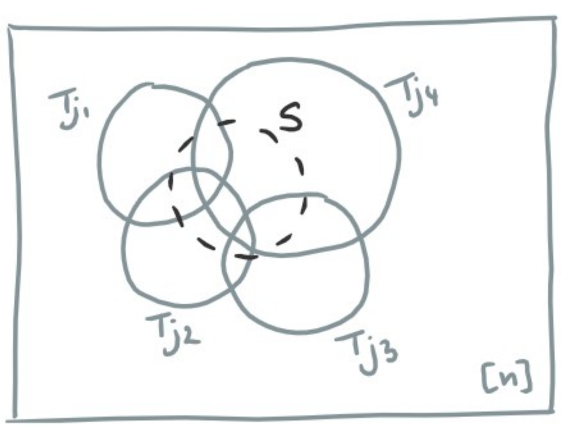
Terms as sets of variables

e.g. $\underbrace{\overline{x_1}}_{T_1} \vee \underbrace{(x_2 \wedge \overline{x_3})}_{T_2} \vee \underbrace{(x_3 \wedge x_4)}_{T_3} \vee \underbrace{(x_3 \wedge \overline{x_5})}_{T_4}$



DNF $\longleftrightarrow$ "family of sets"

Term covers



- Let $u(S)$ = min size of a union of terms covering S.
- Then $|\hat{f}(S)| \leq 2^{-u(S)} \cdot$ (# ways to cover S by a union of size $u(S)$)

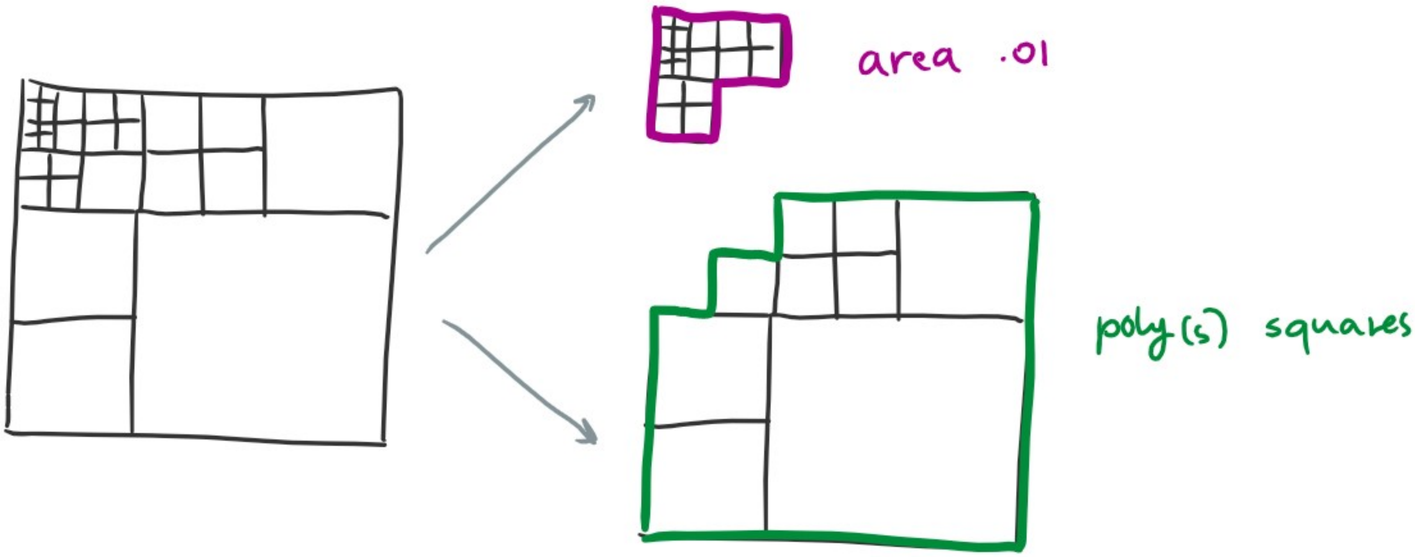
- Then $|\mathcal{F}(S)| \leq 2 \cdot$ (# ways to cover S by a union of size $u(S)$).

- Why it's cool: — specialized for each S
  — completely combinatorial

PART V: OPEN PROBLEMS

Mansour's conjecture

Every size-s DNF .01-concentrated on poly(s) coefficients?



Still wide open for high read!

A concrete case : Talagrand's DNF

Make a DNF by taking the OR of $2^{\sqrt{n}}$ random terms of $\sqrt{n}$ variables (without negations).

of $\sqrt{n}$ variables (without negations).

Is it concentrated on $\mathrm{poly}(s) = 2^{O(\sqrt{n})}$ coefficients?

I'll buy you dinner if you answer this either way! :)

QUESTIONS ?